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Modified Distance Calculation in Generational Distance and Inverted Generational Distance

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Abstract. In this paper, we propose the use of modified distance calculation in generational distance (GD) and inverted generational distance (IGD). These performance indicators evaluate the quality of an obtained solution set in comparison with a pre-specified reference point set. Both indicators are based on the distance between a solution and a reference point. The Euclidean distance in an objective space is usually used for distance calculation. Our idea is to take into account the dominance relation between a solution and a reference point when we calculate their distance. If a solution is dominated by a reference point, the Euclidean distance is used for their distance calculation with no modification. However, if they are non-dominated with each other, we calculate the minimum distance from the reference point to the dominated region by the solution. This distance can be viewed as an amount of the inferiority of the solution (i.e., the insufficiency of its objective values) in comparison with the reference point. We demonstrate using simple examples that some Pareto non-compliant results of GD and IGD are resolved by the modified distance calculation. We also show that IGD with the modified distance calculation is weakly Pareto compliant whereas the original IGD is Pareto non-compliant.

Keywords: Evolutionary multiobjective optimization, performance indicators, generational distance, inverted generational distance, Pareto compliance.

1 Introduction

Evolutionary multiobjective optimization (EMO) has been an active research area in the last two decades [3], [6], [20]. One important issue in this area is performance evaluation of EMO algorithms. Since a set of non-dominated solutions is obtained by a single run of an EMO algorithm, performance evaluation in the EMO community usually means the comparison of different non-dominated solution sets. Various performance indicators have been proposed to evaluate the quality of a non-dominated solution set [8], [14], [15], [26]. Among them, the hypervolume indictor [25] has been most frequently used. This is mainly because no other indicators are Pareto compliant [24]. It has been repeatedly pointed out in the literature [14], [15], [18], [24], [26] that Pareto non-compliant misleading results can be obtained from some other performance indicators.

For example, Zitzler et al. [26] clearly illustrated that misleading results can be obtained from the generational distance (GD) indicator [21] using a simple example of a two-objective minimization problem in Fig. 1 with a reference point set $Z = \{(1, 0), (0, 10)\}$ and three solution sets $A = \{(2, 5)\}$, $B = \{(3, 9)\}$ and $C = \{(10, 10)\}$. GD is the average distance from each solution to its closest reference point. Thus the solution set $B = \{(3, 9)\}$ is evaluated as being the best since (3, 9) has the minimum distance to its nearest reference point among the three solution sets (i.e., *A*, *B* and *C*). However, it is clear from Fig. 1 that $A = \{(2, 5)\}$ is the best among the three solution sets since (3, 9) in *B* and (10, 10) in *C* are dominated by (2, 5) in *A*. A similar example of a two-objective minimization problem was used in Schütze et al. [18], which is shown in Fig. 2 with a reference point set $Z = \{(0, 1), (10, 0)\}$ and two solutions sets $A = \{(5, 2)\}$ and $B = \{(11, 3)\}$. In this example, the solution set *B* is evaluated as being better than the solution set *A* by the GD indicator whereas (11, 3) is dominated by (5, 2).

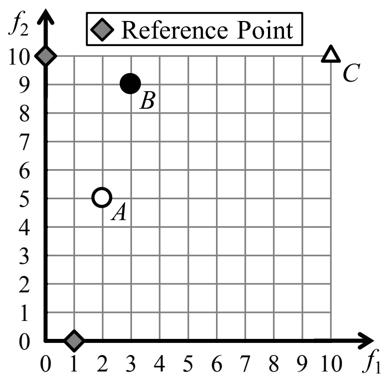


Fig. 1. Example 1 (Zitzler et al. [26]).

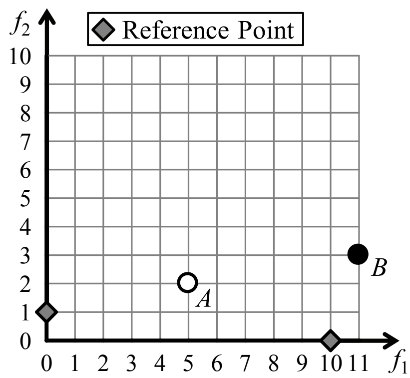


Fig. 2. Example 2 (Schütze et al. [18]).

These misleading results are not obtained by the hypervolume indicator since it is Pareto compliant [24]. One difficulty of the hypervolume indicator is its heavy computation load. Recently evolutionary many-objective optimization has attracted increasing attention [12]. Test problems with ten or more objectives are used for performance evaluation in recent studies on evolutionary many-objective optimization [7], [9], [10], [23]. The use of the hypervolume indicator for those test problems is often impractical from a viewpoint of computation time whereas its fast calculation [17], [22] as well as its efficient approximation [1] has been actively studied. Among other indicators, the inverted generational distance (IGD [4], [19]) is most frequently used for performance evaluation of EMO algorithms in evolutionary many-objective optimization studies [7], [9], [23]. IGD is the average distance from each reference point to its nearest solution. When a set of well-distributed reference points over the entire Pareto front is used, a small value of the IGD indicator suggests the good convergence of solutions to the Pareto front and their good distribution over the entire Pareto front.

In the above-mentioned examples in Fig. 1 and Fig. 2, the solution set *A* is correctly evaluated as being the best by the IGD indicator. Whereas IGD looks a more appropriate indicator than GD in Fig. 1 and Fig. 2, both are Pareto non-compliant. Let us consider another solution set $D = \{(2, 1)\}$ in Fig. 3. It is clear from Fig. 3 that the solution set D is evaluated as being the best among the four solution sets A, B, C and D by the GD indicator. However, the solution set $A = \{(2, 5)\}$ is evaluated as being better than D by the IGD indicator with the Euclidean distance in Fig. 3 as follows:

$$IGD(A) = \frac{1}{2}\left(\sqrt{(2-0)^2+(5-10)^2}+\sqrt{(2-1)^2+(5-0)^2}\right) = 5.24, \quad (1)$$

$$IGD(D) = \frac{1}{2}\left(\sqrt{(2-0)^2+(1-10)^2}+\sqrt{(2-1)^2+(1-0)^2}\right) = 5.32. \quad (2)$$

In Fig. 4, we show another example of a two-objective minimization problem with $Z = \{(0, 10), (1, 6), (2, 2), (6, 1), (10, 0)\}$, $A = \{(2, 4), (3, 3), (4, 2)\}$ and $B = \{(2, 8), (4, 4), (8, 2)\}$. In Fig. 4, each solution in the solution set B is dominated by at least one solution in the solution set A. Thus we can say that A is better than B in the sense of Pareto dominance. The solution set A is also evaluated as being better than B by the GD indicator in Fig. 4. However, if we use the IGD indicator, the solution set B is evaluated as being better than A as follows:

$$IGD(A) = \frac{1}{5}\left(\sqrt{2^2+6^2}+\sqrt{1^2+2^2}+\sqrt{1^2+1^2}+\sqrt{2^2+1^2}+\sqrt{6^2+2^2}\right) = 3.71, \quad (3)$$

$$IGD(B) = \frac{1}{5}\left(\sqrt{2^2+2^2}+\sqrt{1^2+2^2}+\sqrt{2^2+2^2}+\sqrt{2^2+1^2}+\sqrt{2^2+2^2}\right) = 2.59. \quad (4)$$

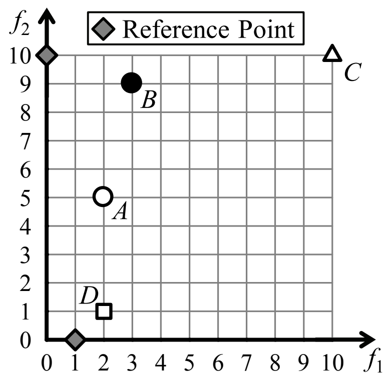


Fig. 3. Example 3 with a new solution set D.

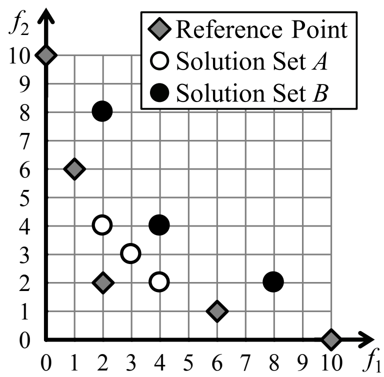


Fig. 4. Example 4 with misleading IGD.

In this paper, first we discuss why these misleading results are obtained by the GD and IGD indicators. Then we propose an idea of modifying the distance calculation between a solution and a reference point in the GD and IGD indicators by taking into account the Pareto dominance relation between them. If a solution is dominated by a reference point, we use the Euclidean distance with no modification. However, if they are non-dominated with each other, we calculate the minimum distance from the reference point to the dominated region by the solution. This distance can be viewed as an amount of the inferiority of the solution (i.e., the insufficiency of its objective values) in comparison with the reference point. Only inferior objective values of the solution to the reference point are used in their distance calculation. In our former study [11], we suggested our idea (i.e., modified distance calculation) as a trick to remedy a severe sensitivity of the IGD indicator to the specification of a reference point set. In this paper, we explain our idea in a more general setting and propose its use in both the GD and IGD indicators. We also show a theoretical property of the IGD measure with the modified distance calculation: weak Pareto compliance.

This paper is organized as follows. In Section 2, we briefly explain multiobjective optimization, Pareto dominance relations, and performance indicators. In Section 3, we explain our idea of modifying the distance calculation in the GD and IGD indicators in detail. In Section 4, we demonstrate that the Pareto non-compliant results in Figs. 1-4 are resolved by the use of the modified distance calculation. Then we show that the IGD indicator with the modified distance calculation is weakly Pareto compliant in Section 5. Finally, we conclude this paper in Section 6.

2 Multiobjective Optimization and Performance Indicators

Let us consider the following m-objective minimization problem with a decision vector $\boldsymbol{x}$ and its feasible region $\boldsymbol{X}$:

$$\text{Minimize } \boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), ..., f_m(\boldsymbol{x})) \text{ subject to } \boldsymbol{x} \in \boldsymbol{X} . \quad (5)$$

In this formulation, $\boldsymbol{z}$ is an m-dimensional objective vector: $\boldsymbol{z} = (z_1, z_2, ..., z_m)$. The feasible region $\boldsymbol{Z}$ of the objective vector $\boldsymbol{z}$ is defined as $\boldsymbol{Z} = \{\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{X}\}$ using the feasible region $\boldsymbol{X}$ of the decision vector $\boldsymbol{x}$.

Let us denote two objective vectors as $\boldsymbol{a} = (a_1, a_2, ..., a_m)$ and $\boldsymbol{b} = (b_1, b_2, ..., b_m)$. They are two points in the m-dimensional objective space. The Pareto dominance relation "$\succ$" and the weak Pareto dominance relation "$\succeq$" are defined for the minimization problem between the two objective vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows:

$$\textbf{Pareto Dominance: } \boldsymbol{a} \succ \boldsymbol{b} \Leftrightarrow \forall i, a_i \le b_i \text{ and } \exists j, a_j < b_j , \quad (6)$$

$$\textbf{Weak Pareto Dominance: } \boldsymbol{a} \succeq \boldsymbol{b} \Leftrightarrow \forall i, a_i \le b_i . \quad (7)$$

The Pareto dominance relation $\boldsymbol{a} \succ \boldsymbol{b}$ means that $\boldsymbol{b}$ is dominated by $\boldsymbol{a}$ (i.e., $\boldsymbol{a}$ is better than $\boldsymbol{b}$). The second condition "$\exists j, a_j < b_j$" in (6) can be replaced with $\boldsymbol{a} \ne \boldsymbol{b}$: $\boldsymbol{a} \succ \boldsymbol{b} \Leftrightarrow \forall i, a_i \le b_i$ and $\boldsymbol{a} \ne \boldsymbol{b}$. The weak Pareto dominance relation $\boldsymbol{a} \succeq \boldsymbol{b}$ means that $\boldsymbol{b}$ is weakly dominated by $\boldsymbol{b}$ (i.e., $\boldsymbol{a}$ is better than or equal to $\boldsymbol{b}$). The weak Pareto dominance relation $\boldsymbol{a} \succeq \boldsymbol{b}$ includes $\boldsymbol{a} = \boldsymbol{b}$ while $\boldsymbol{a} = \boldsymbol{b}$ is excluded from the Pareto dominance relation $\boldsymbol{a} \succ \boldsymbol{b}$.

If an objective vector $z^* = f(x^*)$ is not dominated by any other feasible objective vectors in $\boldsymbol{Z}$, x^* is called a Pareto optimal solution. A set of all Pareto optimal solutions is the Pareto optimal solution set. The projection of the Pareto optimal solution set onto the objective space is called the Pareto front. When x^* is a Pareto optimal solution, $z^* = f(x^*)$ is a Pareto optimal objective vector.

Let A be a set of objective vectors. When no objective vector in A is dominated by any other objective vector in A, A is called a non-dominated set. Let us denote two non-dominated sets of objective vectors as $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_{|A|}\}$ and $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_{|B|}\}$ where $|A|$ and $|B|$ are the cardinality of A and B, respectively.

In Zitzler et al. [26], the Pareto dominance relations between objective vectors were extended to the following relations between objective vector sets (also see [8]):

Pareto Dominance for Sets: $A \succ B \Leftrightarrow \forall \boldsymbol{b}_j \in B, \exists \boldsymbol{a}_i \in A : \boldsymbol{a}_i \succ \boldsymbol{b}_j$, (8)

Weak Pareto Dominance for Sets: $A \succeq B \Leftrightarrow \forall \boldsymbol{b}_j \in B, \exists \boldsymbol{a}_i \in A : \boldsymbol{a}_i \succeq \boldsymbol{b}_j$. (9)

$A \succ B$ and $A \succeq B$ mean that "B is dominated by A" and "B is weakly dominated by A", respectively. $A \succ B$ does not allow the existence of any shared objective vector in A and B. That is, $A \succ B$ requires $(A \cap B) = \phi$. Whereas $A \succ B$ does not allow any overlap between A and B, $A \succeq B$ allows $A = B$ (i.e., A and B can be the same).

In order to handle partially overlapping sets, Zitzler et al. [26] defined an intermediate relation called "better" denoted by "$\triangleright$" as follows (also see [8]):

Relation "better" for Sets: $A \triangleright B \Leftrightarrow A \succeq B$ and $A \neq B$. (10)

This relation $A \triangleright B$ means that A is better than B [26]. The concept of the Pareto compliance [24] of an indicator I(.) can be defined using this relation as follows (it is assumed that a smaller value of the indicator I(.) means a better set):

Pareto Compliant Indicator [24]**:** Whenever $A \triangleright B$ holds between two non-dominated sets A and B, $I(A) < I(B)$ always holds: $A \triangleright B \Rightarrow I(A) < I(B)$.

In this definition, the indicator I(.) is a mapping from a set of objective vectors to a real number. Only the hypervolume is known as being Pareto compliant. In this paper, we also use the following weaker version of the Pareto compliance (some indicators such as the D1 [8] and the unary additive-ε [26] are weakly Pareto compliant):

Weak Pareto Compliant Indicator: Whenever $A \succeq B$ holds between two non-dominated sets A and B, $I(A) \le I(B)$ always holds: $A \succeq B \Rightarrow I(A) \le I(B)$.

As we have already explained in Section 1, the GD and IGD indicators evaluate the quality of an objective vector set using a reference point set. Let $Z = \{z_1, z_2, ..., z_{|Z|}\}$ be a given reference point set where $|Z|$ is the cardinality of Z. The original definition of the GD indicator can be written for a non-dominated objective vector set $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_{|A|}\}$ and the reference point set $Z = \{z_1, z_2, ..., z_{|Z|}\}$ as follows [21]:

Generational Distance: $GD(A) = \frac{1}{|A|}\left(\sum_{i=1}^{|A|} d_i^{\,p}\right)^{1/p}$, (11)

where d_i is the Euclidean distance from $\boldsymbol{a}_i$ to its nearest reference point in Z, and p is an integer parameter. In this paper, we always specify p as $p = 1$ in GD (and IGD).

The IGD indicator is an inverted version of the GD indicator, which is defined as

Inverted Generational Distance: $IGD(A) = \frac{1}{|Z|}\left(\sum_{j=1}^{|Z|} \hat{d}_j^{\,p}\right)^{1/p}$, (12)

where $\hat{d}_j$ is the Euclidean distance from $\boldsymbol{z}_j$ to its nearest objective vector in A.

To the best of our knowledge, the term of "inverted generational distance (IGD)" was first used in 2004 by Coello & Sierra [4] and Sierra & Coello [19]. However, similar indicators had already been used since Czyzak & Jaszkiewicz [5] in 1998. In Czyzak & Jaszkiewicz [5], the weighted achievement scalarizing function was used as the distance between an objective vector and a reference point. Their indicator was denoted as D1 [8], $D1_R$ [14] and I_D [26]. The IGD indicator with the Euclidean distance was used in [2], [13] in 2003 without referring to it as IGD.

Recently, Schütze et al. [18] proposed the following modification of GD and IGD:

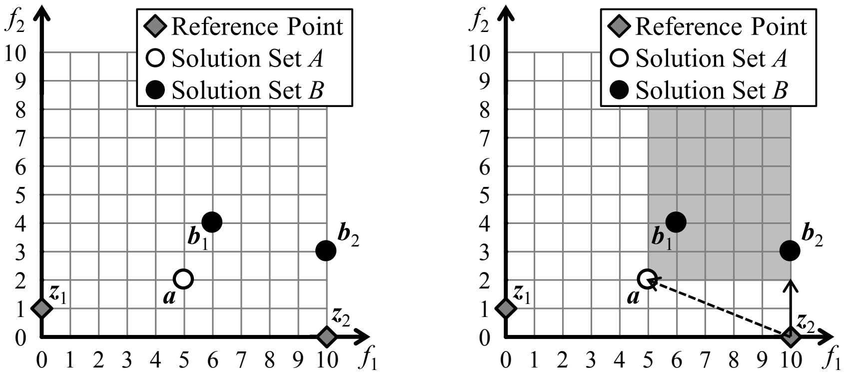
$$GD_p(A) = \left(\frac{1}{|A|}\sum_{i=1}^{|A|} d_i^{\,p}\right)^{1/p} \quad \text{and} \quad IGD_p(A) = \left(\frac{1}{|Z|}\sum_{j=1}^{|Z|} \hat{d}_j^{\,p}\right)^{1/p}. \tag{13}$$

They also proposed a new indicator $\Delta_p(A) = \max\{GD_p(A), IGD_p(A)\}$. This new indicator was used in an indicator-based EMO algorithm in [16].

In this paper, we always specify the value of p as $p = 1$. This is because (i) it makes the meaning of GD and IGD clear, (ii) it has often been used in the literature, and (iii) the modified GD_p and IGD_p [18] in (13) become the same as their original definitions when $p = 1$. The GD with $p = 1$ is the average Euclidean distance from each objective vector to its nearest reference point (and the IGD with $p = 1$ is the average Euclidean distance from each reference point to its nearest objective vector).

3 Modified Distance Calculation

As explained in Section 1, the GD and IGD indicators are Pareto non-compliant. In Fig. 5 (a), we show another example where both GD and IGD are misleading: $GD(A) = 5.10 > GD(B) = 4.33$ and $IGD(A) = 5.24 > IGD(B) = 4.85$. That is, the solution set B is evaluated as being better than the solution set A by the GD and IGD indicators whereas B is dominated by A (i.e., $A \succ B$). These calculations also show that Δ_p is not Pareto compliant since $\Delta_p(A) = 5.24 > \Delta_p(B) = 4.85$ for $p = 1$.



(a) Example 5 with misleading GD and IGD. (b) Distance to the dominated region.

Fig. 5. Example 5 with misleading GD and IGD, and modified distance calculation.

In our former study [11], we suggested an idea to calculate the distance from each reference point to the dominated region by a solution set in the IGD indicator. This idea is illustrated in Fig. 5 (b) where the distance from the reference point $\boldsymbol{z}_2$ to the dominated region by the solution set A is calculated as shown by the vertical solid arrow. The dotted arrow from $\boldsymbol{z}_2$ to $\boldsymbol{a}$ in Fig. 5 (b) shows the standard distance calculation from the reference point $\boldsymbol{z}_2$ to the objective vector $\boldsymbol{a}$.

In this paper, we formulate this idea in a more general setting so that the modified distance calculation can be used in both the GD and IGD indicators. We also explain the motivation behind the modified distance calculation and its meaning in detail.

In Fig. 5, all objective vectors $\boldsymbol{a}$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are dominated by the reference point $\boldsymbol{z}_1$ at (0, 1): $\boldsymbol{z}_1 \succ \boldsymbol{a}$, $\boldsymbol{z}_1 \succ \boldsymbol{b}_1$ and $\boldsymbol{z}_1 \succ \boldsymbol{b}_2$. The two objective vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ in B are also dominated by $\boldsymbol{a}$ in A: $\boldsymbol{a} \succ \boldsymbol{b}_1$ and $\boldsymbol{a} \succ \boldsymbol{b}_2$. In this case, the distance from the reference point $\boldsymbol{z}_1$ is consistent with the Pareto dominance relations among $\boldsymbol{a}$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ as $d(\boldsymbol{z}_1, \boldsymbol{a}) < d(\boldsymbol{z}_1, \boldsymbol{b}_1)$ and $d(\boldsymbol{z}_1, \boldsymbol{a}) < d(\boldsymbol{z}_1, \boldsymbol{b}_2)$ where $d(\boldsymbol{a}, \boldsymbol{b})$ is the Euclidean distance between $\boldsymbol{a}$ and $\boldsymbol{b}$. Actually we can easily prove the following properties:

$$\boldsymbol{z} \succ \boldsymbol{a} \succ \boldsymbol{b} \Rightarrow d(\boldsymbol{z}, \boldsymbol{a}) < d(\boldsymbol{z}, \boldsymbol{b}), \tag{14}$$

$$\boldsymbol{z} \succeq \boldsymbol{a} \succeq \boldsymbol{b} \Rightarrow d(\boldsymbol{z}, \boldsymbol{a}) \le d(\boldsymbol{z}, \boldsymbol{b}). \tag{15}$$

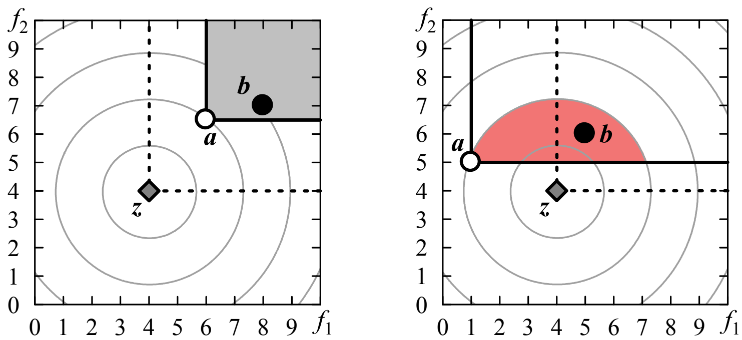
When $\boldsymbol{z} \succ \boldsymbol{a} \succ \boldsymbol{b}$ holds among the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{z}$, we have the following relations for their elements a_i, b_i and z_i (i = 1, 2, ..., m) from $\boldsymbol{z} \succ \boldsymbol{a} \succ \boldsymbol{b}$:

$$\forall i,\ 0 \le a_i - z_i \le b_i - z_i \ \text{ and } \ \exists j,\ 0 \le a_j - z_j < b_j - z_j\,. \tag{16}$$

In this case, $d(\boldsymbol{z}, \boldsymbol{a}) < d(\boldsymbol{z}, \boldsymbol{b})$ always holds. When $\boldsymbol{z} \succeq \boldsymbol{a} \succeq \boldsymbol{b}$ holds, we have the inequality relations "$\forall i,\ 0 \le a_i - z_i \le b_i - z_i$" from the definition of $\boldsymbol{z} \succeq \boldsymbol{a} \succeq \boldsymbol{b}$. In this case, $d(\boldsymbol{z}, \boldsymbol{a}) \le d(\boldsymbol{z}, \boldsymbol{b})$ always holds.

The two properties in (14) and (15) suggest that the GD and IGD indicators can be Pareto compliant under some special conditions. However, the condition $z \succ \boldsymbol{a} \succ \boldsymbol{b}$ does not hold in general as shown by $\boldsymbol{a}$ and z_2 in Fig. 5.

Let us further discuss the distance calculation from a reference point to an objective vector. In Fig. 6, we show contour lines of the Euclidean distance from the reference point z. The shaded region in Fig. 6 (a) shows that all objective vectors $\boldsymbol{b}$ in this region are dominated by $\boldsymbol{a}$. From the contour lines in Fig. 6 (a), we can see that $d(z, \boldsymbol{a}) < d(z, \boldsymbol{b})$ holds for all objective vectors $\boldsymbol{b}$ in the shaded region. This means that the Euclidean distance calculation is consistent with the Pareto dominance relation when $z \succ \boldsymbol{a} \succ \boldsymbol{b}$ holds: $z \succ \boldsymbol{a} \succ \boldsymbol{b} \Rightarrow d(z, \boldsymbol{a}) < d(z, \boldsymbol{b})$. However, in Fig. 6 (b), $\boldsymbol{b}$ has a shorter Euclidean distance than $\boldsymbol{a}$ whereas $\boldsymbol{a} \succ \boldsymbol{b}$ holds. That is, the dominated objective vector $\boldsymbol{b}$ is evaluated as being better than $\boldsymbol{a}$ by the Euclidean distance from z. The contour lines in Fig. 6 (b) show that every objective vector $\boldsymbol{b}$ in the shaded region has a shorter Euclidean distance than $\boldsymbol{a}$ while $\boldsymbol{b}$ is dominated by $\boldsymbol{a}$. This inconsistency can be resolved in Fig. 6 (b) by calculating the minimum distance from z to the shaded area instead of the distance between z and $\boldsymbol{a}$. This modification corresponds to the short vertical arrow in Fig. 5 (b) from z_2 to the dominated region by $\boldsymbol{a}$ (i.e., our modified distance calculation).



(a) Non-dominated region by ***a***.   (b) Inconsistent region of ***b*** for ***a***.

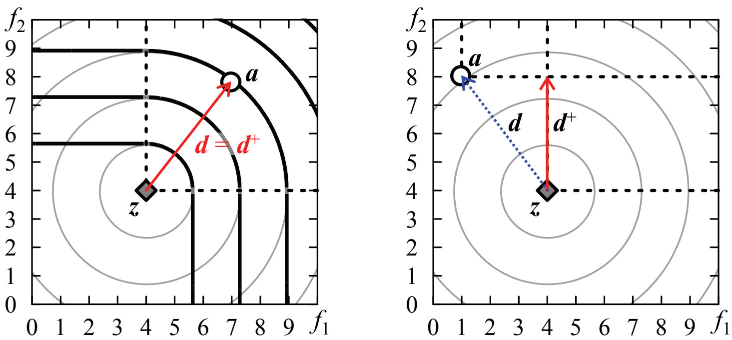
Fig. 6. Contour lines of the Euclidean distance from the reference point z.

In the GD and IGD indicators, smaller values mean better solution sets. The best value of each indicator is zero. Thus the distance $d(z, \boldsymbol{a})$ between the reference point z and the objective vector $\boldsymbol{a}$ used in GD and IGD can be viewed as an error or a penalty to be minimized. The distance can be also interpreted as an amount of the inferiority of $\boldsymbol{a}$ (i.e., the insufficiency of the objective values of $\boldsymbol{a}$) in comparison with z. This interpretation is consistent with the Pareto dominance relation when $\boldsymbol{a}$ is dominated by z as in Fig. 6 (a). In Fig. 6 (a), the decrease in the distance $d(z, \boldsymbol{a})$ by moving $\boldsymbol{a}$ towards z always improves the two objectives of $\boldsymbol{a}$. However, when $\boldsymbol{a}$ and z are non-dominated with each other, the decrease in the distance does not always improve the two objectives of $\boldsymbol{a}$. Actually the move from $\boldsymbol{a}$ towards z in Fig. 6 (b) degrades the first objective whereas it improves the second objective. Moreover, we do not know which is better between $\boldsymbol{a}$ and z since they are non-dominated with each other.

From these discussions, we can see that the distance $d(\boldsymbol{z}, \boldsymbol{a})$ cannot be viewed as an amount of the inferiority to be minimized when $\boldsymbol{a}$ and $\boldsymbol{z}$ are non-dominated with each other. As shown in Fig. 7, the distance $d(\boldsymbol{z}, \boldsymbol{a})$ is the length of the vector $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{z}$. Each element d_i of $\boldsymbol{d}$ (i.e., $d_i = a_i - z_i$) shows how a_i is inferior to (i.e., larger than) z_i with respect to the ith objective. Thus a positive value of d_i can be viewed as an amount of the inferiority (i.e., insufficiency) of a_i to z_i. However, if d_i is negative, a_i is superior to (i.e., smaller than) z_i. In this case, a negative values of d_i is viewed as having no inferiority (i.e., no insufficiency). As a result, we define an inferiority (i.e., insufficiency) vector $\boldsymbol{d}^+ = (d_1^+, d_2^+, ..., d_m^+)$ as follows:

$$d_i^+ = \max\{a_i - z_i, 0\}, \ i = 1, 2, ..., m. \tag{17}$$

When $\boldsymbol{z} \succ \boldsymbol{a}$ holds, $\boldsymbol{d}^+$ is the same as $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{z}$ (see Fig. 7 (a)). However, when $\boldsymbol{z} \succ \boldsymbol{a}$ does not hold, $\boldsymbol{d}^+$ is different from $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{z}$ since only the positive elements of $\boldsymbol{d}$ remain in $\boldsymbol{d}^+$. In Fig. 7 (b), the vector $\boldsymbol{d}^+$ is shown by the solid vertical arrow together with the dotted arrow $\boldsymbol{d}$. It should be noted that the definition in (17) is replaced with $d_i^+ = \max\{z_i - a_i, 0\}$ for multiobjective maximization problems.



(a) When $\boldsymbol{z} \succ \boldsymbol{a}$ holds. (b) When $\boldsymbol{z} \succ \boldsymbol{a}$ does not hold.

Fig. 7. Illustration of the two vectors $\boldsymbol{d}$ and $\boldsymbol{d}^+$.

Using (17), we propose the use of the following modified distance calculation $d^+(\boldsymbol{z}, \boldsymbol{a})$ in the GD and IGD indicators instead of the Euclidean distance $d(\boldsymbol{z}, \boldsymbol{a})$:

Modified Distance Calculation for Minimization Problems:

$$d^+(\boldsymbol{z}, \boldsymbol{a}) = \sqrt{d_1^{+2} + \cdots + d_m^{+2}} = \sqrt{(\max\{a_1 - z_i, 0\})^2 + \cdots + (\max\{a_m - z_m, 0\})^2}. \tag{18}$$

Modified Distance Calculation for Maximization Problems:

$$d^+(\boldsymbol{z}, \boldsymbol{a}) = \sqrt{d_1^{+2} + \cdots + d_m^{+2}} = \sqrt{(\max\{z_1 - a_1, 0\})^2 + \cdots + (\max\{z_m - a_m, 0\})^2}. \tag{19}$$

Contour lines of the modified distance from $\boldsymbol{z}$ are shown by solid bold lines in Fig. 7 (a). In the right upper region of $\boldsymbol{z}$ where $\boldsymbol{z} \succ \boldsymbol{a}$, the modified distance is the same as the Euclidean distance. However, in the left upper region of $\boldsymbol{z}$ in Fig. 7 (a), the contour lines are horizontal parallel straight lines since only a_2 is used (and vertical parallel straight lines since only a_1 is used in the right lower region of $\boldsymbol{z}$).

In this paper, we denote the GD and IGD indicators with the modified distance calculation in (18) by GD$^+$ and IGD$^+$, respectively. This is because only the positive elements of $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{z}$ are used in the distance calculation in (18).

The vector $\boldsymbol{d}^+$ defined by (17) can be viewed as showing the minimum amount of the increase from $\boldsymbol{z}$ so that $\boldsymbol{z} + \boldsymbol{d}^+$ is weakly dominated by the objective vector $\boldsymbol{a}$. Let us assume that $\boldsymbol{z} + \boldsymbol{u}$ is weakly dominated by $\boldsymbol{a}$. That is, $a_i \le z_i + u_i$ for all i's. If $\boldsymbol{z} \succ \boldsymbol{a}$ holds, $\boldsymbol{u}$ with the minimum length is obtained from $u_i = a_i - z_i$ for all i's. However, if $\boldsymbol{z}$ and $\boldsymbol{a}$ are non-dominated with each other, there exists at least a pair of a_i and z_i with $a_i < z_i$. Such a z_i does not have to be increased from its current value. Thus $u_i = 0$ if $0 > a_i - z_i$. For the other a_i's with $0 \le a_i - z_i$, u_i is specified as $u_i = a_i - z_i$ so that $a_i \le z_i + u_i$ holds with the minimum increase u_i. This definition of $\boldsymbol{u}$ is the same as the definition of $\boldsymbol{d}^+$ in (17). In Fig. 7 (a), the move of $\boldsymbol{d}$ is needed to make $\boldsymbol{z} + \boldsymbol{d}$ be weakly dominated by $\boldsymbol{a}$. However, the move of $\boldsymbol{d}$ is not needed in Fig. 7 (b). This is because $\boldsymbol{z} + \boldsymbol{d}^+$ is dominated by $\boldsymbol{a}$ in Fig. 7 (b).

These discussions can be summarized as the following minimization problem of the Euclidean norm $\|\boldsymbol{u}\|$, which can explain $\boldsymbol{d}^+$ in (17) and $d^+(\boldsymbol{z}, \boldsymbol{a})$ in (18):

$$\text{Minimize } \|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_m^2} \text{ subject to } \boldsymbol{a} \succeq \boldsymbol{z} + \boldsymbol{u}. \tag{20}$$

The vector $\boldsymbol{d}^+$ defined in (17) is the optimal solution $\boldsymbol{u}^*$ of this problem. The modified distance $d^+(\boldsymbol{z}, \boldsymbol{a})$ in (18) is the corresponding optimal value. The standard Euclidean distance corresponds to the optimal value of (20) with the equality constraint $\boldsymbol{a} = \boldsymbol{z} + \boldsymbol{u}$ instead of the weak dominance constraint. For multiobjective maximization problems, the constraint $\boldsymbol{a} \succeq \boldsymbol{z} + \boldsymbol{u}$ in (20) is replaced with $\boldsymbol{a} \succeq \boldsymbol{z} - \boldsymbol{u}$ where $\boldsymbol{u}$ shows the decrease from $\boldsymbol{z}$ so that $\boldsymbol{z} - \boldsymbol{u}$ can be weakly dominated by the objective vector $\boldsymbol{a}$. The optimal value of the minimization problem of $\|\boldsymbol{u}\|$ with $\boldsymbol{a} \succeq \boldsymbol{z} - \boldsymbol{u}$ corresponds to the modified distance calculation in (19).

4 Effects of Modified Distance Calculation

We examine the effects of the modified distance calculation in (18) using the six examples in Figs. 1-5 (i.e., Examples 1-5) and Fig. 6 (b). The value of p is always specified as p = 1. In Tables 1-6, we show the values of GD, IGD, GD$^+$ and IGD$^+$ together with the dominance relation among the given solution sets. It should be noted that GD$_p$ and IGD$_p$ [18] with $p = 1$ are the same as GD and IGD, respectively.

In each table, a better indicator value (i.e., smaller value) is highlighted by bold. These tables show that all Pareto non-compliant results are removed by the modified distance calculation. However, the GD^+ indicator is not Pareto compliant as shown in Fig. 8 and Table 7. In Fig. 8, the solution set A dominates the solution set B. However, B is evaluated as being better than A by GD and GD^+ in Table 7.

The IGD^+ indicator is consistent with the Pareto dominance relation (i.e., if $A \succeq B$ holds, the inconsistent result $IGD^+(A) > IGD^+(B)$ is not obtained). This property will be explained in the next section. However, the IGD^+ indicator is not Pareto compliant in the strict sense as shown in Fig. 9 and Table 8. In Fig. 9, the solution set A is better than the solution set B (i.e., $A \triangleright B$). However, in Table 8, $IGD^+(A) < IGD^+(B)$ does not hold. Actually, $I(A) = I(B)$ holds for IGD and IGD^+. This is because the nearest objective vector (2, 2) from the reference point $z = (0, 0)$ is shared by the two solution sets A and B in Fig. 9: A = {(1, 8), (2, 2), (8, 1)}, B = {(2, 2)} and Z = {(0, 0)}.

Table 1. Example 1 in Fig. 1 ($A \succ B$).

Indicator	$I(A)$	$I(B)$	$I(A) < I(B)$
GD	5.099	**3.162**	Inconsistent
GD^+	**2.000**	3.000	OK
IGD	**5.242**	6.191	OK
IGD^+	**3.550**	6.110	OK

Table 2. Example 2 in Fig. 2 ($A \succ B$).

Indicator	$I(A)$	$I(B)$	$I(A) < I(B)$
GD	5.099	**3.162**	Inconsistent
GD^+	**2.000**	3.162	OK
IGD	**5.242**	7.171	OK
IGD^+	**3.550**	7.171	OK

Table 3. Example 3 in Fig. 3 ($D \succ A$).

Indicator	$I(D)$	$I(A)$	$I(D) < I(A)$
GD	**1.414**	5.099	OK
GD^+	**1.414**	2.000	OK
IGD	5.317	**5.242**	Inconsistent
IGD^+	**1.707**	3.550	OK

Table 4. Example 4 in Fig. 4 ($A \succ B$).

Indicator	$I(A)$	$I(B)$	$I(A) < I(B)$
GD	**1.805**	2.434	OK
GD^+	**1.138**	2.276	OK
IGD	3.707	**2.591**	Inconsistent
IGD^+	**1.483**	2.260	OK

Table 5. Example 5 in Fig. 5 ($A \succ B$).

Indicator	$I(A)$	$I(B)$	$I(A) < I(B)$
GD	5.099	**4.328**	Inconsistent
GD^+	**2.000**	3.500	OK
IGD	5.242	**4.854**	Inconsistent
IGD^+	**3.550**	4.854	OK

Table 6. Example in Fig. 6 (b) ($A \succ B$).

Indicator	$I(A)$	$I(B)$	$I(A) < I(B)$
GD	3.162	**2.236**	Inconsistent
GD^+	**1.000**	2.236	OK
IGD	3.162	**2.236**	Inconsistent
IGD^+	**1.000**	2.236	OK

Table 7. Example in Fig. 8 ($A \succ B$).

Indicator	$I(A)$	$I(B)$	$I(A) < I(B)$
GD	6.318	**5.000**	Inconsistent
GD^+	6.318	**5.000**	Inconsistent
IGD	**2.828**	5.000	OK
IGD^+	**2.828**	5.000	OK

Table 8. Example in Fig. 9 ($A \triangleright B$).

Indicator	$I(A)$	$I(B)$	$I(A) < I(B)$
GD	6.318	**2.828**	Inconsistent
GD^+	6.318	**2.828**	Inconsistent
IGD	2.828	2.828	$I(A) = I(B)$
IGD^+	2.828	2.828	$I(A) = I(B)$

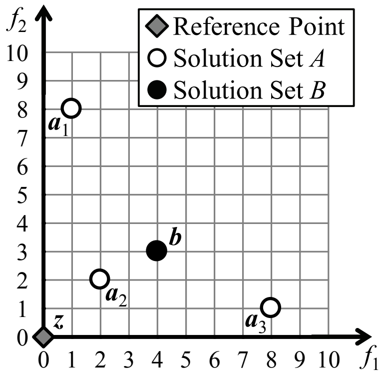


Fig. 8. Example with misleading GD$^+$.

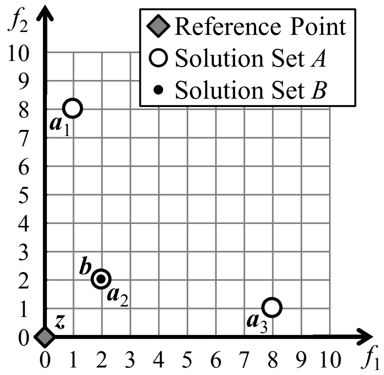


Fig. 9. Example with overlapping solutions.

5 Weak Pareto Compliance of the IGD$^+$ Indicator

In this section, we show that the IGD$^+$ indicator is weakly Pareto compliant (i.e., $A \succeq B \Rightarrow I(A) \leq I(B)$). That is, we show that $IGD^+(A) \leq IGD^+(B)$ always holds whenever $A \succeq B$ holds between two non-dominated sets *A* and *B*. It should be noted that the IGD$^+$ indicator is not Pareto compliant in the strict sense: Even when $A \rhd B$ holds, $IGD^+(A) < IGD^+(B)$ does not always hold (see Fig. 9 and Table 8 where $IGD^+(A) = IGD^+(B)$ and $A \rhd B$).

Before showing the weak Pareto compliance property of the IGD$^+$ indicator, we first show that $d^+(\boldsymbol{z}, \boldsymbol{a}) \leq d^+(\boldsymbol{z}, \boldsymbol{b})$ always holds whenever $\boldsymbol{a} \succeq \boldsymbol{b}$ holds. From the definition of the weak Pareto dominance $\boldsymbol{a} \succeq \boldsymbol{b}$, we have $\forall i,\ a_i \leq b_i$. Thus we have $\forall i,\ a_i - z_i \leq b_i - z_i$. Then the following relation is obtained:

$$\forall i,\ 0 \ \leq\ d_i^+(\boldsymbol{z}, \boldsymbol{a}) = \max\{a_i - z_i, 0\} \ \leq\ \max\{b_i - z_i, 0\} = d_i^+(\boldsymbol{z}, \boldsymbol{b})\,. \qquad (21)$$

From (21), we can see that $d^+(\boldsymbol{z}, \boldsymbol{a}) \leq d^+(\boldsymbol{z}, \boldsymbol{b})$ holds. That is, $d^+(\boldsymbol{z}, \boldsymbol{a}) \leq d^+(\boldsymbol{z}, \boldsymbol{b})$ always holds whenever $\boldsymbol{a} \succeq \boldsymbol{b}$ holds:

$$\boldsymbol{a} \succeq \boldsymbol{b} \quad \Rightarrow \quad d^+(\boldsymbol{z}, \boldsymbol{a}) \leq d^+(\boldsymbol{z}, \boldsymbol{b})\,. \qquad (22)$$

It should be noted that this relation holds for an arbitrarily specified reference point ***z*** since we do not use any assumption on ***z***. For example, (22) holds when ***z*** is non-dominated with ***a*** and ***b***. It also holds even when ***z*** is dominated by ***a*** and ***b***.

Using (22), we show that the IGD$^+$ indicator is weakly Pareto compliant. Let us assume that $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_{|A|}\}$ and $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_{|B|}\}$ are non-dominated sets where

$A \succeq B$ holds. We also assume that $Z = \{z_1, z_2, ..., z_{|Z|}\}$ is a non-dominated reference point set. From the definition of the weak Pareto dominance relation between the two sets A and B (i.e., $A \succeq B$), the following relation holds:

$$\forall \boldsymbol{b}_j \in B, \exists \boldsymbol{a}_i \in A : \boldsymbol{a}_i \succeq \boldsymbol{b}_j . \tag{23}$$

$IGD^+(B)$ is calculated in the following manner. First the distance from each reference point $\boldsymbol{z}_k$ to the nearest objective vector in B is calculated using the modified distance calculation $d^+(\boldsymbol{z}, \boldsymbol{b})$. Then the average value is calculated over all reference points in Z. Let $\boldsymbol{b}_{j(k)}$ be the nearest objective vector in B to $\boldsymbol{z}_k$ with respect to the modified distance calculation $d^+(\boldsymbol{z}, \boldsymbol{b})$ where $j(k) \in \{1, 2, ..., |B|\}$ and $k = 1, 2, ..., |Z|$. The distance from each $\boldsymbol{z}_k$ in Z to its nearest objective vector $\boldsymbol{b}_{j(k)}$ in B is $d^+(\boldsymbol{z}_k, \boldsymbol{b}_{j(k)})$. Thus $IGD^+(B)$ is calculated as

$$IGD^+(B) = \frac{1}{|Z|} \sum_{k=1}^{|Z|} d^+(\boldsymbol{z}_k, \boldsymbol{b}_{j(k)}) . \tag{24}$$

From (23), there exists at least one $\boldsymbol{a}_{i(j(k))}$ in A that satisfies $\boldsymbol{a}_{i(j(k))} \succeq \boldsymbol{b}_{j(k)}$ for each $\boldsymbol{b}_{j(k)}$ where $i(j(k)) \in \{1, 2, ..., |A|\}$ and $k = 1, 2, ..., |Z|$. That is, we can choose $\boldsymbol{a}_{i(j(k))}$ for each $\boldsymbol{b}_{j(k)}$ for $k = 1, 2, ..., |Z|$ in (24) such that $\boldsymbol{a}_{i(j(k))} \succeq \boldsymbol{b}_{j(k)}$. From (22), we have

$$\boldsymbol{a}_{i(j(k))} \succeq \boldsymbol{b}_{j(k)} \quad \Rightarrow \quad d^+(\boldsymbol{z}_k, \boldsymbol{a}_{i(j(k))}) \le d^+(\boldsymbol{z}_k, \boldsymbol{b}_{j(k)}) . \tag{25}$$

Since $\boldsymbol{a}_{i(j(k))} \succeq \boldsymbol{b}_{j(k)}$ holds for $k = 1, 2, ..., |Z|$, $d^+(\boldsymbol{z}_k, \boldsymbol{a}_{i(j(k))}) \le d^+(\boldsymbol{z}_k, \boldsymbol{b}_{j(k)})$ also holds for $k = 1, 2, ..., |Z|$. Thus we obtain the following inequality relation:

$$IGD^+(A) \le \frac{1}{|Z|} \sum_{k=1}^{|Z|} d^+(\boldsymbol{z}_k, \boldsymbol{a}_{i(j(k))}) \le \frac{1}{|Z|} \sum_{k=1}^{|Z|} d^+(\boldsymbol{z}_k, \boldsymbol{b}_{j(k)}) = IGD^+(B) . \tag{26}$$

The first inequality in (26) holds since the distance from $\boldsymbol{z}_k$ to its nearest objective vector in A is equal to or smaller than $d^+(\boldsymbol{z}_k, \boldsymbol{a}_{i(j(k))})$. When $\boldsymbol{a}_{i(j(k))}$ is the nearest objective vector in A to $\boldsymbol{z}_k$ for all k's, the equality holds between the first two terms in (26). The second inequality in (26) holds from (25).

6 Conclusions

In this paper, we proposed the use of the modified distance calculation instead of the Euclidean distance in the GD and IGD indicators. The Pareto dominance relation between a reference point and an objective vector is taken into account in our modified distance calculation. Using simple numerical examples of two-objective minimization problems, we demonstrated that some Pareto non-compliant results of the GD and IGD indicators are resolved by the use of our modified distance calculation. We also showed that the IGD indicator with our modified distance calculation, which is called the IGD$^+$ indicator, is weakly Pareto compliant whereas IGD is Pareto non-compliant. One advantage of IGD$^+$ over the frequently-used hypervolume indictor is its computational efficiency. No heavy computation is added to IGD in the modified distance calculation. That is, a theoretical property is added to IGD in the IGD$^+$ indicator with no severe increase in its computation load.

As shown in this paper, the Pareto compliant property between two objective vectors (i.e., $\boldsymbol{a} \succ \boldsymbol{b} \Rightarrow d(\boldsymbol{z}, \boldsymbol{a}) < d(\boldsymbol{z}, \boldsymbol{b})$) does not always hold when the Euclidean distance is used. That is, if a reference point $\boldsymbol{z}$ and an objective vector $\boldsymbol{a}$ are non-dominated with each other, an inconsistent result $d(\boldsymbol{z}, \boldsymbol{a}) > d(\boldsymbol{z}, \boldsymbol{b})$ can be obtained for $\boldsymbol{a}$ and $\boldsymbol{b}$ with $\boldsymbol{a} \succ \boldsymbol{b}$. This inconsistency leads to Pareto non-compliant results of the GD and IGD indicators. However, when our modified distance calculation is used, the weak Pareto compliant property always holds: $\boldsymbol{a} \succeq \boldsymbol{b} \Rightarrow d^+(\boldsymbol{z}, \boldsymbol{a}) \le d^+(\boldsymbol{z}, \boldsymbol{b})$. Good results of GD$^+$ and IGD$^+$ were obtained from this property. Especially, it was shown that the IGD$^+$ indicator is weakly Pareto compliant.

One may feel some similarity between our modified distance calculation and the epsilon indicator [26]. In its additive version, ε is used for all elements of all reference points. That is, the maximum distance over all reference points (and over all objectives of each reference point) is calculated instead of the average distance in IGD$^+$. One may also feel some similarity between our modified distance calculation and the weighted achievement scalarizing function used in the D1 indicator [8]. In IGD$^+$, the Euclidean distance is usually used as shown in Fig. 7 (a).

Future research topics include theoretical and experimental studies on the effects of the modified distance calculation on evaluation results by the GD and IGD indicators of EMO algorithms. Only a few experimental results were reported in [11]. It may be an interesting study to re-evaluate recently reported performance evaluation results of EMO algorithms on many-objective problems using the IGD$^+$ indicator.

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